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Structural Eigenderivative Analysis Using Practical and Simplified Dynamic Flexibility Method

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A practical and simplified technique for computing a large number of eigenvector derivatives of a complex structural system has been developed. This technique uses the dynamic flexibility method and is based on the practical complete modal space approach. The practical complete modal space technique is an exact modal method. There is no modal truncation error existing in the derivation. The dynamic flexibility method requires only the solving of the system governing equation once, regardless of the number of eigenvector derivatives needed to be computed. Using this method, the computer time required to obtain more than one eigenvector derivative can be dramatically reduced. This method gives the mathematical expression of the solution for eigenvector derivatives and is easier for engineers to use to perform theoretical formulation. This method can be applied to systems with and without repeated eigenvalues. This method gives better numerical precision and may be a very good tool for engineers to compute many eigenvector derivatives.

Nomenclature

$A_0, A_1, \ldots$	= matrices to be determined
C	= scalar constant matrix to be determined in general solution of Z'
$F(\lambda) \in R^{n,n}$	= practical accurate dynamic flexibility matrix
$\tilde{F}(\lambda) \in R^{n,n}$	= accurate interim dynamic flexibility matrix
$F_s(\lambda) \in R^{n,n}$	= simplified approximate dynamic flexibility matrix
$\tilde{F}_s(\lambda) \in R^{n,n}$	= simplified interim dynamic flexibility matrix
g	= number of eigenvalues except m-multiple eigenvalue λ in Ω
h	= number of higher-order eigenvectors
$K \in R^{n,n}$	= positive semidefinite stiffness matrix
$K' \in R^{n,n}$	= derivative of stiffness matrix K
k	= number of lower-order eigenvectors
$M \in R^{n,n}$	= positive definite mass matrix
$M' \in R^{n,n}$	= derivative of mass matrix M
m	= number of eigenvalues contained in repeated root λ
n	= dimension (degrees of freedom) of eigensystem
r	= number of eigenvalues except m-multiple eigenvalue λ in Ω_k
$S(\lambda) \in R^{n,n}$	= dynamic stiffness matrix
X	= arbitrary square matrix
$Z \in R^{n,m}$	= eigenvector matrix of λ
$Z' \in R^{n,m}$	= derivative of eigenvector matrix Z
Z'_{dp}	= direct perturbed solution of Z'
Z'_{op}	= original perturbed solution of Z'
Z'_{ps}	= particular solution of Z'
η	= nondimensional perturbation value
κ_h	= generalized stiffness matrix of Ψ_h
Λ'	= derivative diagonal matrix of λ
λ	= m-multiple eigenvalue
μ_h	= generalized mass matrix of Ψ_h
$\Phi \in R^{n,n}$	= complete eigenvector matrix
$\Phi_h(\lambda) \in R^{n,h}$	= higher-order eigenvector matrix
$\Phi_k(\lambda) \in R^{n,k}$	= lower-order eigenvector matrix
φ_i	= ith column vector of Φ
$\Psi_h(\lambda) \in R^{n,h}$	= equivalent higher-order eigenvector matrix
$\Omega \in R^{n,n}$	= complete eigenvalue diagonal matrix
$\Omega_h(\lambda) \in R^{h,h}$	= higher-order eigenvalue diagonal matrix
$\Omega_k(\lambda) \in R^{k,k}$	= lower-order eigenvalue diagonal matrix
$\omega_1^2, \omega_2^2, \ldots$	= 1st, 2nd, ... diagonal element of Ω or Ω_k

Superscripts

T	= transpose of matrix
$+$	= generalized inverse of matrix

I. Introduction

COMPUTATION of eigenvector derivatives for a large complex aerospace structure is a time-consuming job. This situation is especially critical when one has to find an optimal design solution for a given large, complex aerospace structural dynamic system. This is because a given aerospace structural system requires the engineer to compute a large number of eigenvector derivatives repeatedly to achieve the best design goal; therefore, finding a new approach to reduce computer time is an essential and challenging assignment for many design engineers.

There are several useful methods published in the literature describing how to find eigenvector derivatives of an eigensystem.[1–12] These methods can be classified into two different types of approaches: 1) direct computational method[1–9] and 2) indirect computational method.[10–12]

All of the published methods require solving a set of system governing equations with respect to the eigenvalue of interest for each eigenvector derivative computation. The coefficient matrix of the system governing equation needs to be decomposed once for each different eigenvalue. Such a technique is tremendously time consuming when the structural dynamics engineer needs to design a complex system with many eigenvector derivatives to be computed.

In an optimal structural design, calculation of a large number of eigenvector derivatives is often required during the design process. To reduce the computer time, a new method is presented using the dynamic flexibility technique combined with the practical complete modal space to compute eigenvector derivatives. The dynamic flexibility technique only needs to analyze the system coefficient matrix once, regardless of the number of the eigenvector derivatives

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needed. Thus, this method shows high computational efficiency as compared to other methods when one needs to compute a large number of eigenvector derivatives.

Also, the practical complete modal space technique used is an exact modal method. There is no modal truncation error existing in the derivation. In the practical complete modal space method, the lower-order modes, including rigid-body modes, are obtained from solving the eigenequation. All higher-order modes are replaced by equivalent higher-order modes that are given by using a simple matrix projection approach.

Both theoretical derivation and numerical results are presented to verify the analysis. Using this method, the computer time required to obtain more than one eigenvector derivative is dramatically reduced. The method gives the mathematical expression of the solution for eigenvector derivatives and is easier for engineers to use to perform theoretical formulation. The amount of computer time saved during the design can be directly obtained from the computation when compared to other published methods. This method is a very good tool for engineers to compute many eigenvector derivatives to optimize a complex aerospace structural dynamic system during the design.

II. Technical Approach

A. Practical Accurate Dynamic Flexibility Formula

In a structural dynamic analysis, the dynamic stiffness and dynamic flexibility matrices, respectively, can be defined as

$$S(\lambda) = K - \lambda M \tag{1}$$

$$F(\lambda) = (K - \lambda M)^{-1} = S^{-1}(\lambda) \tag{2}$$

in which K is a positive semidefinite matrix and M is a positive definite matrix. The eigensystem of interest can have either a single root or multiple roots. Here λ is considered as m-multiple roots. Eigenvectors corresponding to λ are defined as $Z \in R^{n,m}$. Also define $\Phi = [\varphi_1, \varphi_2, \ldots, \varphi_n]$ and $\Omega = \text{diag}[\omega_1^2, \omega_2^2, \ldots, \omega_n^2]$ as the complete eigenpair matrices of the system obtained from the eigenequation

$$(K - \omega_i^2 M)\varphi_i = 0 \tag{3}$$

From the mass normalized condition one knows $\Phi^{-1} = \Phi^T M$; thus, K and M can be represented as

$$\begin{aligned} K &= M\Phi\Omega\Phi^T M = M\left[\sum_{i=1}^{n} \omega_i^2 \varphi_i \varphi_i^T\right] M \\ M &= M\Phi\Phi^T M = M\left[\sum_{i=1}^{n} \varphi_i \varphi_i^T\right] M \end{aligned} \tag{4}$$

Substituting Eq. (4) into Eq. (1) gives

$$S(\lambda) = M\Phi(\Omega - \lambda I)\Phi^T M = M\left[\sum_{i=1}^{n} (\omega_i^2 - \lambda)\varphi_i \varphi_i^T\right] M \tag{5}$$

Because $\text{rank}(S(\lambda)) = n - m = g$, there are only g nonzero terms in Eq. (5). Thus, the inverse matrix of Eq. (2) does not exist; one can find the generalized inverse $S^+(\lambda)$ as

$$S^+(\lambda) = F(\lambda) = \Phi(\Omega - \lambda I)^+ \Phi^T \tag{6}$$

If one defines $(\Omega - \lambda I)^+$ as

$$(\Omega - \lambda I)^+ = \begin{bmatrix} (\omega_1^2 - \lambda)^{-1} & & & & \vdots & \\ & (\omega_2^2 - \lambda)^{-1} & & & \vdots & 0 \\ & & \ddots & & \vdots & \\ & & & (\omega_g^2 - \lambda)^{-1} & \vdots & \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ & & 0 & & \vdots & \\ & & & & \vdots & 0 \\ & & & & \vdots & (m, m) \end{bmatrix} \tag{7}$$

then $S^+(\lambda)$ can be proven to satisfy two Moore–Penrose generalized inverse relationships:

$$SS^+S = S, \qquad S^+SS^+ = S^+ \tag{8}$$

Here, Φ can be divided into two subgroups: lower-order mode set $\Phi_k \in R^{n,k}$ and higher-order mode set $\Phi_h \in R^{n,h}$. If λ is a frequency of interest falling inside the range $\omega_1^2 \le \lambda \le \omega_k^2$, then ω_1 is the lowest frequency and ω_k is the highest frequency existing in the Φ_k, and Eq. (6) can be rewritten as

$$F(\lambda) = \Phi_k(\Omega_k - \lambda I)^+ \Phi_k^T + \Phi_h(\Omega_h - \lambda I)^{-1} \Phi_h^T \tag{9}$$

where

$$(\Omega_k - \lambda I)^+ = \begin{bmatrix} (\omega_1^2 - \lambda)^{-1} & & & & \vdots & \\ & (\omega_2^2 - \lambda)^{-1} & & & \vdots & 0 \\ & & \ddots & & \vdots & \\ & & & (\omega_r^2 - \lambda)^{-1} & \vdots & \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ & & 0 & & \vdots & \\ & & & & \vdots & 0 \\ & & & & \vdots & (m, m) \end{bmatrix} \tag{10}$$

where subscript $r = g - h = k - m$ and $k + h = n$.

An equivalent higher-order mode Ψ_h of Φ_h has been introduced in this paper. The process for forming Ψ_h is described briefly as follows: First compute

$$\bar{\Psi}_h = \Phi_h \Phi_h^T M = I - \Phi_k \Phi_k^T M, \qquad (\bar{\Psi}_h \in R^{n,n}) \tag{11}$$

Obviously, $\text{rank}(\bar{\Psi}_h) = h = n - k$. This indicates that there are only h independent columns in the matrix $\bar{\Psi}_h$. These h independent columns are nonuniquely determined in the $\bar{\Psi}_h$ matrix because there are always k dependent columns in the matrix. From the mathematical point of view, one can adopt any numerical technique to eliminate k dependent columns from the matrix $\bar{\Psi}_h$ to obtain Ψ_h ($\in R^{n,h}$). The equivalence between Ψ_h and Φ_h has been proven in Refs. 13 and 14, and it is demonstrated that

$$\Phi_k^T K \Psi_h = 0, \qquad \Phi_k^T M \Psi_h = 0 \tag{12}$$

$$\Psi_h^T K \Psi_h = \kappa_h \ne \Omega_h \text{ or diagonal matrix} \tag{13a}$$

$$\Psi_h^T M \Psi_h = \mu_h \ne I \text{ or diagonal matrix} \tag{13b}$$

The aforementioned equivalence implies that the entire contribution of Ψ_h is equivalent to that of Φ_h. One can deduce that the entire modal potential energy and kinetic energy of both Ψ_h and Φ_h must be equal. The entire potential energy and the kinetic energy of Φ_h are expressed as the sum of elements from the matrices $\Phi_h^T K \Phi_h = \Omega_h$ and $\Phi_h^T M \Phi_h = I$, respectively. Now, the corresponding energies of Ψ_h in the analysis are expressed as the sum of elements of matrices κ_h and μ_h. Thus, under the mass normalized condition, there exists

$$\sum_{i=1}^{h} \sum_{j=1}^{h} (\kappa_h)_{ij} = \sum_{i=1}^{h} (\Omega_h)_{ii} \tag{14a}$$

$$\sum_{i=1}^{h} \sum_{j=1}^{h} (\mu_h)_{ij} = \sum_{i=1}^{h} (I)_{ii} \tag{14b}$$

According to the preceding statement, Φ_h and its modal energies can be replaced by Ψ_h and its corresponding modal energies, respectively, that is, Φ_h, Ω_h, and I shown in the second term of Eq. (9) can be separately replaced by Ψ_h, κ_h, and μ_h. Thus, Eq. (9) can be rewritten as

$$F(\lambda) = \Phi_k(\Omega_k - \lambda I)^+ \Phi_k^T + \Psi_h(\kappa_h - \lambda \mu_h)^{-1} \Psi_h^T \tag{15}$$

From Eq. (10), Eq. (15) can also be rewritten as

$$F(\lambda) = \bar{\Phi}_k(\bar{\Omega}_k - \lambda I)^{-1}\bar{\Phi}_k^T + \Psi_h(\kappa_h - \lambda\mu_h)^{-1}\Psi_h^T \qquad (16)$$

in which $\bar{\Omega}_k$ and $\bar{\Phi}_k$ are the eigenpair matrices that do not include m-multiple eigenvalue λ and the corresponding eigenvectors Z. Equations (15) and (16) are the practical accurate dynamic flexibility formulas.

B. Simplified Approximate Dynamic Flexibility Formula

Equation (15) can be used in comparison with Eq. (9), but the calculation of the inverse matrix as shown in Eq. (15) is still time consuming because the inverse matrix is a function of λ. To avoid the calculation of the inverse, the matrix inverse in the second term of Eq. (15) can be expressed as a power series of λ as

$$(\kappa_h - \lambda\mu_h)^{-1} \cong A_0 + \lambda A_1 + \lambda^2 A_2 + \cdots \qquad (17)$$

where $A_0, A_1, \ldots$ are constant matrices to be determined. Equation (17) can also be expressed as

$$(\kappa_h - \lambda\mu_h)(A_0 + \lambda A_1 + \lambda^2 A_2 + \cdots) = I \qquad (18)$$

Expanding Eq. (18), collecting all terms with the same order of λ in both sides of Eq. (18), and then setting these terms equal gives

$$\lambda^0: \quad \kappa_h A_0 = I \qquad (19)$$

$$\lambda^1: \quad \kappa_h A_1 - \mu_h A_0 = 0 \qquad (20a)$$

$$\lambda^2: \quad \kappa_h A_2 - \mu_h A_1 = 0 \qquad (20b)$$

$$\lambda^3: \quad \kappa_h A_3 - \mu_h A_2 = 0 \qquad (20c)$$

$$\vdots \qquad \vdots$$

From Eq. (20) one obtains

$$\kappa_h A_p = \mu_h A_{p-1}, \qquad p = 1, 2, \ldots \qquad (21)$$

Obviously, Eqs. (19) and (21) show that for all matrices $A_0, A_1, \ldots$ the coefficient matrix κ_h only needs to be decomposed once regardless of the number of eigenvector derivatives needed to be computed.

Substituting Eq. (17) into Eq. (15) gives the simplified approximate dynamic flexibility formula

$$F_s(\lambda) = \Phi_k(\Omega_k - \lambda I)^{+}\Phi_k^T + \Psi_h A_0 \Psi_h^T + \lambda\Psi_h A_1 \Psi_h^T + \lambda^2\Psi_h A_2 \Psi_h^T + \cdots \qquad (22a)$$

$$= \bar{\Phi}_k(\bar{\Omega}_k - \lambda I)^{-1}\bar{\Phi}_k^T + \Psi_h A_0 \Psi_h^T + \lambda\Psi_h A_1 \Psi_h^T + \lambda^2\Psi_h A_2 \Psi_h^T + \cdots \qquad (22b)$$

To explain the convergence of the power series shown in Eq. (17), Eqs. (19) and (21) are rewritten as

$$A_0 = \kappa_h^{-1} \qquad (23a)$$

$$A_p = \kappa_h^{-1}\mu_h A_{p-1}, \qquad p = 1, 2, \ldots \qquad (23b)$$

Embedding Eq. (23) into Eq. (17) yields

$$(\kappa_h - \lambda\mu_h)^{-1} \cong \kappa_h^{-1} + \lambda\kappa_h^{-1}\mu_h\kappa_h^{-1} + \lambda^2\kappa_h^{-1}\mu_h\kappa_h^{-1}\mu_h\kappa_h^{-1} + \cdots \qquad (24)$$

In addition, there is a known mathematical formula

$$(I - X)^{-1} = I + X + X^2 + \cdots, \qquad \|X\| < 1 \qquad (25)$$

When one assumes $X = \lambda\kappa_h^{-1}\mu_h$, Eq. (24) can be obtained from Eq. (25). Thus, it is shown that the power series existing in Eq. (22) is truly convergent if the Euclidean norm $\|\lambda\kappa_h^{-1}\mu_h\| < 1$. When the value of λ increases so that $\|\lambda\kappa_h^{-1}\mu_h\| > 1$, Eq. (22) cannot be used to find the solution. Under this circumstance, a shifting frequency technique can be utilized. However, for the $\|\lambda\kappa_h^{-1}\mu_h\| > 1$ condition the authors have developed a different simplified dynamic flexibility formula as follows.

When the $\|\lambda\kappa_h^{-1}\mu_h\| > 1$ condition is true, the inverse matrix $(\kappa_h - \lambda\mu_h)^{-1}$ in the second term of Eq. (15) is expanded as an inversed power series of λ as

$$(\kappa_h - \lambda\mu_h)^{-1} = A_0 + \lambda^{-1}A_{-1} + \lambda^{-2}A_{-2} + \cdots \qquad (26)$$

Require

$$(\kappa_h - \lambda\mu_h)(A_0 + \lambda^{-1}A_{-1} + \lambda^{-2}A_{-2} + \cdots) - I = 0 \qquad (27)$$

Expanding Eq. (27), collecting all terms with the same order of λ, and then setting these terms equal to zero yields

$$\lambda^1: \quad \mu_h A_0 = 0 \qquad (28)$$

$$\lambda^0: \quad \mu_h A_{-1} - \kappa_h A_0 + I = 0 \qquad (29)$$

$$\lambda^{-1}: \quad \mu_h A_{-2} - \kappa_h A_{-1} = 0 \qquad (30)$$

$$\lambda^{-2}: \quad \mu_h A_{-3} - \kappa_h A_{-2} = 0 \qquad (31)$$

$$\vdots \qquad \vdots$$

Obviously, from Eq. (28) one knows $A_0 = 0$. Substituting the equation $A_0 = 0$ into Eq. (29) gives

$$\mu_h A_{-1} = -I \qquad (32)$$

Combining Eqs. (30) and (31) finds

$$\mu_h A_{-p} = \kappa_h A_{-(p-1)}, \qquad p \geq 2 \qquad (33)$$

After matrices $A_0, A_{-1}, A_{-2}, \ldots$ are obtained using Eqs. (28), (32), and (33), embedding Eq. (26) into Eq. (15) yields

$$F_s(\lambda) = \Phi_k(\Omega_k - \lambda I)^{+}\Phi_k^T + \lambda^{-1}\Psi_h A_{-1}\Psi_h^T + \lambda^{-2}\Psi_h A_{-2}\Psi_h^T + \cdots \qquad (34)$$

If the negative sign on the right-hand side of Eq. (32) is changed to a positive sign, that is, let Eq. (32) become

$$\mu_h A_{-1} = I \qquad (35)$$

then Eq. (34) can be rewritten as

$$F_s(\lambda) = \Phi_k(\Omega_k - \lambda I)^{+}\Phi_k^T - \lambda^{-1}\Psi_h A_{-1}\Psi_h^T - \lambda^{-2}\Psi_h A_{-2}\Psi_h^T - \cdots \qquad (36)$$

Using Eqs. (28), (32), and (33), Eq. (26) can be rewritten as

$$(\kappa_h - \lambda\mu_h)^{-1} = -\lambda^{-1}\mu_h^{-1} - \lambda^{-2}\mu_h^{-1}\kappa_h\mu_h^{-1} - \lambda^{-3}\mu_h^{-1}\kappa_h\mu_h^{-1}\kappa_h\mu_h^{-1} - \cdots \qquad (37)$$

In addition, $(\kappa_h - \lambda\mu_h)^{-1}$ can be rewritten as

$$-\lambda^{-1}(I - \lambda^{-1}\mu_h^{-1}\kappa_h)^{-1}\mu_h^{-1}$$

in which $(I - \lambda^{-1}\mu_h^{-1}\kappa_h)^{-1}$ is expanded using Eq. (25) with the expression $X = \lambda^{-1}\mu_h^{-1}\kappa_h$. Equation (37) can be found from Eq. (25). This indicates that the inversed power series shown in Eq. (34) or (36) is truly convergent if the norm $\|\lambda^{-1}\mu_h^{-1}\kappa_h\| < 1$. Clearly, when $\|\lambda\kappa_h^{-1}\mu_h\| > 1$, the inequality $\|\lambda^{-1}\mu_h^{-1}\kappa_h\| < 1$ normally exists. Note that Eq. (34) or (36) is just a required simplified dynamic flexibility formula that is suitable to the condition $\|\lambda\kappa_h^{-1}\mu_h\| > 1$.

III. Computation of Derivatives for Many Eigenvectors

In this section, the computation of derivatives for many eigenvectors with repeated eigenvalues is addressed. The calculation of the eigenvector derivative for a single root condition is only treated as a special case in the analysis.

The governing equation for eigenvector derivative Z' of repeated root λ is

$$(K - \lambda M)Z' = MZ\Lambda' - (K' - \lambda M')Z = G \qquad (38)$$

where Λ' is a diagonal matrix that consists of derivatives λ_i' $(i = 1, 2, \ldots, m)$ of λ. K' and M' are the derivatives of K and M, respectively. Because the coefficient matrix $(K - \lambda M)$ is singular, Eq. (38) is an indeterminate equation. To eliminate the singularity, a direct perturbation technique[7,8] is adopted to perturb the parameter λ. Thus, Eq. (38) becomes

$$[K - (1+\eta)\lambda M]Z'_{op} = G \tag{39}$$

in which η is a small quantity without dimension. In general, the value of η is approximately 0.001. This number is obtained based on the authors' experience. The solution of Z'_{op} can be expressed as

$$Z'_{op} = F_s(\tilde{\lambda})G \tag{40}$$

Here $\tilde{\lambda} = (1+\eta)\lambda$ and Z'_{op} is defined as the original perturbed (OP) solution. From Eq. (22) one knows

$$F_s(\tilde{\lambda}) = [K - (1+\eta)\lambda M]^{-1} = \Phi_k[\Omega_k - (1+\eta)\lambda I]^{-1}\Phi_k^T + \sum_{p=0}^{s}(1+\eta)^p\lambda^p\Psi_h A_p\Psi_h^T \tag{41a}$$

$$F_s(\tilde{\lambda}) = \bar{\Phi}_k[\bar{\Omega}_k - (1+\eta)\lambda I]^{-1}\bar{\Phi}_k^T - \eta^{-1}\lambda^{-1}ZZ^T + \sum_{p=0}^{s}(1+\eta)^p\lambda^p\Psi_h A_p\Psi_h^T \tag{41b}$$

For the reader's convenience, the power series of λ shown in Eq. (22) is rewritten using a summation notation, in which s is the number of the terms determined on the basis of the required precision. Note that, for the $\|\lambda\kappa_h^{-1}\mu_h\| > 1$ condition, $F_s(\tilde{\lambda})$ shall be as follows [see Eq. (34)]:

$$F_s(\tilde{\lambda}) = \Phi_k[\Omega_k - (1+\eta)\lambda I]^{-1}\Phi_k^T + \sum_{p=1}^{s}(1+\eta)^{-p}\lambda^{-p}\Psi_h A_{-p}\Psi_h^T \tag{42a}$$

$$F_s(\tilde{\lambda}) = \bar{\Phi}_k[\bar{\Omega}_k - (1+\eta)\lambda I]^{-1}\bar{\Phi}_k^T - \eta^{-1}\lambda^{-1}ZZ^T + \sum_{p=0}^{s}(1+\eta)^{-p}\lambda^{-p}\Psi_h A_{-p}\Psi_h^T \tag{42b}$$

The existence of the second term shown in Eq. (41b) or (42b) is because the eigenvector of Z is included in the Φ_k. To make the solution of Eq. (41b) or (42b) approach more closely Eq. (22b), the second term shown in Eq. (41b) or (42b) has to be removed. Sometimes this term is not a small value. Based on the mass orthonormal condition, left multiplying both sides of Eq. (41b) or (42b) by ZZ^TM yields

$$ZZ^TMF_s(\tilde{\lambda}) = -\eta^{-1}\lambda^{-1}ZZ^T \tag{43}$$

Subtracting Eq. (43) from Eq. (41b) or (42b) gives the simplified perturbed dynamic flexibility formula

$$\bar{F}_s(\tilde{\lambda}) = (I - ZZ^TM)F_s(\tilde{\lambda}) \tag{44}$$

Clearly, $\bar{F}_s(\tilde{\lambda})$ is an approximation solution. In comparison with $F_s(\tilde{\lambda})$, $\bar{F}_s(\tilde{\lambda})$ is closer to $F_s(\lambda)$. Thus, the better eigenvector derivative equation is expressed as

$$Z'_{dp} = \bar{F}_s(\tilde{\lambda})G \tag{45}$$

Here Z'_{dp} is defined as the direct perturbed (DP) solution.

The Z'_{dp} must be considered as a particular solution Z'_{ps} of Eq. (38). The general solution of the eigenvector derivatives is[3–6]

$$Z' = Z'_{ps} + ZC \tag{46}$$

The scalar constant matrix C is found by the procedure shown in Refs. 4 and 6.

Note that using the practical perturbed dynamic flexibility formula $\bar{F}(\tilde{\lambda})$, as shown in Eqs. (48) and (49), Z'_{dp} solutions can be found from

$$Z'_{dp} = \bar{F}(\tilde{\lambda})G \tag{47}$$

in which

$$\bar{F}(\tilde{\lambda}) = (I - ZZ^TM)F(\tilde{\lambda}) \tag{48}$$

$$F(\tilde{\lambda}) = \Phi_k[\Omega_k - (1+\eta)\lambda I]^{-1}\Phi_k^T + \Psi_h[\kappa_h - (1+\eta)\lambda\mu_h]^{-1}\Psi_h^T \tag{49}$$

Finally, note that $A_p (p \geq 0)$ and $A_{-p} (p \geq 1)$ shown in Eqs. (41) and (42) only need to be computed once for any given λ because A_p and A_{-p} are not a function of λ. Equation (44) is very efficient in the calculation of many eigenvector derivatives. In addition, once the development of practical complete modal space as shown in Refs. 13 and 14 makes κ_h and μ_h become diagonal matrices, the practical dynamic flexibility method shown in Eqs. (47–49) will be one of the best methods. Equations (47–49) can give high-precision eigenvector derivatives, and the calculation of the inverse matrix $[\kappa_h - (1+\eta)\lambda\mu_h]^{-1}$ in the second term of Eq. (49) is relatively easy.

IV. Evolution of Generalized Inverse Technique

The generalized inverse technique described in Ref. 9 is deduced here so that different practical dynamic flexibility and simplified dynamic flexibility formulas can be obtained. To remove the singularity of the coefficient matrix $S(\lambda)$ of Eq. (38), the following nonsingular matrix $\tilde{S}(\lambda)$ has been established (shown in Ref. 9):

$$\tilde{S}(\lambda) = S(\lambda) + MZDZ^TM \tag{50}$$

where $D = \text{diag}[d_1, d_2, \ldots, d_m]$ and d_j $(j = 1, 2, \ldots, m)$ are arbitrary nonzero scalar constants. In addition, the relationship

$$\tilde{F}(\lambda) = \tilde{S}^{-1}(\lambda) = S^+(\lambda) + ZD^{-1}Z^T \tag{51}$$

can be found in Ref. 9. $\tilde{F}(\lambda)$ is an accurate interim dynamic flexibility matrix acquired by the generalized inverse technique. One can give an interim particular solution $\tilde{Z}'_{ps}$ of the eigenvector derivative Z' as

$$\tilde{S}(\lambda)\tilde{Z}'_{ps} = G \tag{52}$$

From Eqs. (51) and (52), $\tilde{Z}'_{ps}$ can be expressed as

$$\tilde{Z}'_{ps} = \tilde{F}(\lambda)G \tag{53a}$$

$$\tilde{Z}'_{ps} = Z'_{ps} + ZD^{-1}Z^TG \tag{53b}$$

Here the definition of Z'_{ps} is

$$Z'_{ps} = S^+(\lambda)G \tag{54}$$

From Eq. (53b), the final particular solution obtained using the generalized inverse technique[9] is

$$Z'_{ps} = \tilde{Z}'_{ps} - ZD^{-1}Z^TG \tag{55}$$

As with other methods, the generalized inverse technique used in Ref. 9 is unsuitable for the calculations of derivatives of many eigenvectors because Eq. (52) needs to be solved for each different λ value. However, by the use of the practical complete modal space,[13,14] different practical dynamic flexibility and simplified dynamic flexibility formulas can be found using the generalized inverse technique.[9] From both Eqs. (6) and (9), Eq. (51) can be rewritten as

$$\tilde{F}(\lambda) = \Phi_k(\Omega_k - \lambda I)^+\Phi_k^T + \Phi_h(\Omega_h - \lambda I)^{-1}\Phi_h^T + ZD^{-1}Z^T \tag{56}$$

As with Eq. (15), Eq. (56) can be given as

$$\tilde{F}(\lambda) = \Phi_k(\Omega_k - \lambda I)^{+}\Phi_k^T + \Psi_h(\kappa_h - \lambda\mu_h)^{-1}\Psi_h^T + ZD^{-1}Z^T \tag{57}$$

Equation (57) is a different form of the practical dynamic flexibility formula as shown in this paper. Similarly, from Eq. (57) another simplified dynamic flexibility formula is given as follows:

$$\tilde{F}_s(\lambda) = \Phi_k(\Omega_k - \lambda I)^{+}\Phi_k^T + ZD^{-1}Z^T + \Psi_h A_0 \Psi_h^T + \lambda\Psi_h A_1 \Psi_h^T + \lambda^2 \Psi_h A_2 \Psi_h^T + \cdots \tag{58}$$

By the use of either Eq. (57) or Eq. (58), the interim particular solution $\tilde{Z}'_{ps}$ is first computed using Eq. (53a). Then the final particular solution can be found from Eq. (55).

V. Numerical Example

The numerical example (a cantilever beam with 12 degrees of freedom) is constructed by adding an additional beam element to the example used in Ref. 4. It is a uniform beam with a square cross section resulting in out-of-plane and in-plane bending modes with the same frequencies. The stiffness and mass matrix formulas of the beam element can be found in Ref. 4. The design parameters of the beam elements are as follows: Young's modulus $E = 1000$, length $L = 1$, cross-sectional area $A = 420$, mass density $\rho = 1$, and area moments of inertia $I_y = I_z = 1$ for the y and z axes. Thus, the assemble stiffness and mass matrices, K and M, of the cantilever beam are, respectively,

$$K = \begin{bmatrix}
12 & 0 & 0 & 6 & -12 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\
 & 12 & -6 & 0 & 0 & -12 & -6 & 0 & 0 & 0 & 0 & 0 \\
 & & 4 & 0 & 0 & 6 & 2 & 0 & 0 & 0 & 0 & 0 \\
 & & & 4 & -6 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
 & & & & 24 & 0 & 0 & 0 & -12 & 0 & 0 & 6 \\
 & & & & & 24 & 0 & 0 & 0 & -12 & -6 & 0 \\
 & & & & & & 8 & 0 & 0 & 6 & 2 & 0 \\
 & & & & & & & 8 & -6 & 0 & 0 & 2 \\
 & & & \text{symmetry} & & & & & 24 & 0 & 0 & 0 \\
 & & & & & & & & & 24 & 0 & 0 \\
 & & & & & & & & & & 8 & 0 \\
 & & & & & & & & & & & 8
\end{bmatrix} \times 10^3 \tag{59}$$

$$M = \begin{bmatrix}
156 & 0 & 0 & 22 & 54 & 0 & 0 & -13 & 0 & 0 & 0 & 0 \\
 & 156 & -22 & 0 & 0 & 54 & 13 & 0 & 0 & 0 & 0 & 0 \\
 & & 4 & 0 & 0 & -13 & -3 & 0 & 0 & 0 & 0 & 0 \\
 & & & 4 & 13 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\
 & & & & 312 & 0 & 0 & 0 & 54 & 0 & 0 & -13 \\
 & & & & & 312 & 0 & 0 & 0 & 54 & 13 & 0 \\
 & & & & & & 8 & 0 & 0 & -13 & -3 & 0 \\
 & & & & & & & 8 & 13 & 0 & 0 & -3 \\
 & & \text{symmetry} & & & & & & 312 & 0 & 0 & 0 \\
 & & & & & & & & & 312 & 0 & 0 \\
 & & & & & & & & & & 8 & 0 \\
 & & & & & & & & & & & 8
\end{bmatrix} \tag{60}$$

A design parameter, with respect to which the eigenvectors are differentiated, is the area moment of inertia I_z of the beam element at the free end. Therefore the derivatives K' and M' of matrices K and M are, respectively,

$$K' = \begin{bmatrix}
12 & 0 & 0 & -6 & -12 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 4 & -6 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
 & & & & 12 & 0 & 0 & -6 & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & 4 & 0 & 0 & 0 & 0 \\
 & & & & & & & & 0 & 0 & 0 & 0 \\
 & & & \text{symmetry} & & & & & & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 \\
 & & & & & & & & & & & 0
\end{bmatrix} \times 10^3 \tag{61}$$

and $M' = 0$. Also, there are $K'' = M'' = 0$. This cantilever beam has six 2-multiple roots. The first to the fourth repeated eigenvalues are $\lambda_1 = 0.36346$, $\lambda_2 = 14.365$, $\lambda_3 = 114.70$, and $\lambda_4 = 581.67$,

respectively. The eigenvectors for λ_1–λ_4 are Z_1–Z_4, respectively, and are

$$Z_1 = \begin{bmatrix} 0 & 0.056355 \\ 0.056355 & 0 \\ 0.025858 & 0 \\ 0 & -0.025858 \\ 0 & 0.030823 \\ 0.030823 & 0 \\ 0.024588 & 0 \\ 0 & -0.024588 \\ 0 & 0.009329 \\ 0.009329 & 0 \\ 0.016991 & 0 \\ 0 & -0.016991 \end{bmatrix} \tag{62a}$$

$$Z_2 = \begin{bmatrix} 0 & -0.056692 \\ -0.056692 & 0 \\ -0.090423 & 0 \\ 0 & 0.090423 \\ 0 & 0.024006 \\ 0.024006 & 0 \\ -0.055925 & 0 \\ 0 & 0.055925 \\ 0 & 0.033441 \\ 0.033441 & 0 \\ 0.033327 & 0 \\ 0 & -0.033327 \end{bmatrix}$$

$$Z_3 = \begin{bmatrix} 0 & 0.056615 \\ 0.056615 & 0 \\ 0.15001 & 0 \\ 0 & -0.15001 \\ 0 & -0.037145 \\ -0.037145 & 0 \\ -0.023941 & 0 \\ 0 & 0.023941 \\ 0 & 0.042124 \\ 0.042124 & 0 \\ -0.031364 & 0 \\ 0 & 0.031364 \end{bmatrix} \tag{62b}$$

$$Z_4 = \begin{bmatrix} 0 & -0.060920 \\ 0.060920 & 0 \\ 0.25963 & 0 \\ 0 & 0.25963 \\ 0 & 0.007090 \\ -0.007090 & 0 \\ -0.20089 & 0 \\ 0 & -0.20089 \\ 0 & 0.015346 \\ -0.015346 & 0 \\ 0.19608 & 0 \\ 0 & 0.19608 \end{bmatrix}$$

The derivatives of λ_1–λ_4 are

$$\lambda_{1,1} = 0, \qquad \lambda_{1,2} = 0.0027581, \qquad \lambda_{2,1} = 0$$
$$\lambda_{2,2} = 1.8694, \qquad \lambda_{3,1} = 0, \qquad \lambda_{3,2} = 41.588 \tag{63}$$
$$\lambda_{4,1} = 0, \qquad \lambda_{4,2} = 230.0$$

The eigenvector derivatives of λ_1–λ_4 are computed by using Eqs. (41), (42), and (49). Computational cases included are listed as follows:

1) Φ_k is constructed by using eight eigenvectors of the first four 2-multiple eigenvalues, λ_1–λ_4. The simplified dynamic flexibility (SDF) formulas of Eqs. (41) and (42) are used to compute the eigenvector derivatives. Taking the power series of λ^0–λ^1 terms, λ^0–λ^2 terms, and λ^0–λ^3 terms from Eq. (41), as well as taking the λ^{-1} term, λ^{-1}–λ^{-2} terms, and λ^{-1}–λ^{-3} terms from Eq. (42) to perform calculation of eigenvector derivatives, this is considered as the first-order, the second-order, and the third-order approximate solutions of eigenvector derivatives.

2) Φ_k is constructed by two different eigenvectors of the first double root λ_1, and the practical dynamic flexibility (PDF) formula of Eq. (49) is used to compute the eigenvector derivatives.

The numerical results are listed in Tables 1–4, in which the exact values are found through the procedures shown in Ref. 7. That is, the exact values are obtained by using the following formulas:

$$[K - (1+\eta)\lambda M]Z'_{\text{op}} = G \tag{64a}$$

$$Z'_{\text{dp}} = (I - ZZ^T M)Z'_{\text{op}} = Z'_{\text{ps}} \tag{64b}$$

$$Z'_e = Z'_{\text{ps}} + ZC \tag{64c}$$

In Tables 1–4, $NM1$, $NM2$, and $Z'_{ij}\%$ are defined as follows, respectively:

$$NM1 = \|\lambda\kappa_h^{-1}\mu_h\|, \qquad NM2 = \|\lambda^{-1}\mu_h^{-1}\kappa_h\| \tag{65}$$

$$Z'_{ij}\% = \|Z'_{e,ij} - Z'_{ij}\|/\|Z'_{e,ij}\|\% \tag{66}$$

in which $Z'_{e,ij}$ is an exact value and Z'_{ij} represents the results obtained by using the PDF and SDF methods in this paper. Because $\|\lambda\kappa_h^{-1}\mu_h\| < 1$, i.e., $\|\lambda^{-1}\mu_h^{-1}\kappa_h\| > 1$, for the first to the fourth 2-multiple eigenvalues, the precision of the results from Eq. (42) is poor, and results from Eq. (42) are not listed in Tables 1–4. To prove the truth of Eq. (42), derivatives of the two highest-order modes (eigenvectors of the fifth to the sixth 2-multiple roots) have to be computed. Equation (42) cannot be utilized. For this reason, the variety [Eq. (A7)] of Eq. (42) is given in the Appendix. The eigenvector derivatives of the fifth and the sixth repeated roots λ_5 and λ_6 are obtained by using Eqs. (A7) and (A8) and are listed in Tables 5 and 6. Here $NM3$ and $NM4$ are defined as

$$NM3 = \|\lambda^{-1}\mu_k^{-1}\kappa_k\|, \qquad NM4 = \|\lambda\kappa_k^{-1}\mu_k\| \tag{67}$$

The derivatives of eigenvalues λ_5 and λ_6 are as follows:

$$\lambda'_{5,1} = 0, \qquad \lambda'_{5,2} = 491.93$$
$$\lambda'_{6,1} = 0, \qquad \lambda'_{6,2} = 7087.1 \tag{68}$$

From the results shown in Tables 1–6, there are several important observations, which are listed as follows:

1) In the subset of eigenvector that satisfies the $\|\lambda\kappa_h^{-1}\mu_h\| < 1$ condition, the eigenvectors are formed by the first double root's eigenvectors Z_1 to the fourth double root's eigenvectors Z_4. The numerical precision of derivatives of the higher-order eigenvectors is poorer than that of derivatives of lower-order eigenvectors. The larger the λ value is, the slower is the convergence speed of the λ power series, as shown in Eq. (41). Also, in the subset of eigenvector that satisfies the $\|\lambda^{-1}\mu_h^{-1}\kappa_n\| < 1$ condition, the precision of derivatives of the higher-order eigenvectors is better than that of the lower-order eigenvectors (see Tables 5 and 6). The larger the λ value is, the faster is the convergence speed of the λ power series, as shown in Eqs. (42) and (A7). However, the closer to one the $\|\lambda\kappa_h^{-1}\mu_h\|$ and $\|\lambda^{-1}\mu_h^{-1}\kappa_h\|$ (or $\|\lambda^{-1}\mu_K^{-1}\kappa_K\|$) values are, the poorer is the numerical

Table 1 Eigenvector derivatives of the first repeated root ($NM1 = 0.257 \times 10^{-3}$, $NM2 = 0.329 \times 10^{5}$)

				SDF method based on Eq. (41)					
Exact value		PDF method		First-order approximation		Second-order approximation		Third-order approximation	
$Z'_{1.1}$	$Z'_{1.2}$	$Z'_{1.1}$	$Z'_{1.2}$	$Z'_{1.1}$	$Z'_{1.2}$	$Z'_{1.1}$	$Z'_{1.2}$	$Z'_{1.1}$	$Z'_{1.2}$
0.0	−0.52410D−3	0.0	−0.52410D−3	0.0	−0.52410D−3	0.0	−0.52410D−3	0.0	−0.52410D−3
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	0.10865D−2	0.0	0.10865D−2	0.0	0.10865D−2	0.0	0.10865D−2	0.0	0.10865D−2
0.0	0.23736D−3	0.0	0.23736D−3	0.0	0.23736D−3	0.0	0.23736D−3	0.0	0.23736D−3
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	−0.17973D−3	0.0	−0.17973D−3	0.0	−0.17973D−3	0.0	−0.17973D−3	0.0	−0.17973D−3
0.0	0.74750D−4	0.0	0.74750D−4	0.0	0.74750D−4	0.0	0.74750D−4	0.0	0.74750D−4
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	−0.13341D−3	0.0	−0.13341D−3	0.0	−0.13341D−3	0.0	−0.13341D−3	0.0	−0.13341D−3
Z'_{ij}%		0	0	0	0.4D−6	0	0.5D−9	0	0.5D−9

Table 2 Eigenvector derivatives of the second repeated root ($NM1 = 0.0102$, $NM2 = 0.831 \times 10^{3}$)

				SDF method based on Eq. (41)					
Exact value		PDF method		First-order approximation		Second-order approximation		Third-order approximation	
$Z'_{2.1}$	$Z'_{2.2}$	$Z'_{2.1}$	$Z'_{2.2}$	$Z'_{2.1}$	$Z'_{2.2}$	$Z'_{2.1}$	$Z'_{2.2}$	$Z'_{2.1}$	$Z'_{2.2}$
0.0	0.66634D−2	0.0	0.66643D−2	0.0	0.66643D−2	0.0	0.66643D−2	0.0	0.66643D−2
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	−0.19374D−1	0.0	−0.19374D−1	0.0	−0.19374D−1	0.0	−0.19374D−1	0.0	−0.19374D−1
0.0	−0.35262D−2	0.0	−0.35262D−2	0.0	−0.35262D−2	0.0	−0.35262D−2	0.0	−0.35262D−2
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	0.12701D−1	0.0	0.12701D−1	0.0	0.12701D−1	0.0	0.12701D−1	0.0	0.12701D−1
0.0	0.29117D−2	0.0	0.29117D−2	0.0	0.29117D−2	0.0	0.29117D−2	0.0	0.29117D−2
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	−0.70129D−3	0.0	−0.70129D−3	0.0	−0.70141D−3	0.0	−0.70129D−3	0.0	−0.70129D−3
Z'_{ij}%		0	0	0	0.85D−3	0	0.6D−5	0	0.6D−6

Table 3 Eigenvector derivatives of the third repeated root ($NM1 = 0.0812$, $NM2 = 0.104 \times 10^{3}$)

				SDF method based on Eq. (41)					
Exact value		PDF method		First-order approximation		Second-order approximation		Third-order approximation	
$Z'_{3.1}$	$Z'_{3.2}$	$Z'_{3.1}$	$Z'_{3.2}$	$Z'_{3.1}$	$Z'_{3.2}$	$Z'_{3.1}$	$Z'_{3.2}$	$Z'_{3.1}$	$Z'_{3.2}$
0.0	−0.80777D−2	0.0	−0.80777D−2	0.0	−0.80692D−2	0.0	−0.80772D−2	0.0	−0.80778D−2
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	0.43668D−1	0.0	0.43668D−1	0.0	0.43613D−1	0.0	0.43665D−1	0.0	0.43668D−1
0.0	−0.15341D−2	0.0	−0.15341D−2	0.0	−0.15301D−2	0.0	−0.15339D−2	0.0	−0.15342D−2
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	−0.61715D−1	0.0	−0.61715D−1	0.0	−0.61685D−1	0.0	−0.61714D−1	0.0	−0.61715D−1
0.0	0.12031D−2	0.0	0.12031D−2	0.0	0.12007D−2	0.0	0.12030D−2	0.0	0.12031D−2
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	0.25943D−1	0.0	0.25943D−1	0.0	0.25987D−1	0.0	0.25945D−1	0.0	0.25942D−1
Z'_{ij}%		0	0	0	0.096	0	0.0055	0	0.5D−3

precision of the derivatives of the corresponding eigenvectors. For the eigenvectors with $\|\lambda\kappa_h^{-1}\mu_h\|$ and $\|\lambda^{-1}\mu_h^{-1}\kappa_h\|$ that are close to one, the eigenvector derivatives of the fifth repeated root (Table 5), a shifting frequency technique can obviously improve the precision of the derivatives. Note that the advent of the $\|\lambda\kappa_h^{-1}\mu_h\| > 1$ and $\|\lambda\kappa_h^{-1}\mu_h\| \cong 1$ conditions is rare.

2) For $\|\lambda^{-1}\mu_h^{-1}\kappa_h\| < 1$, i.e., $\|\lambda\kappa_h^{-1}\mu_h\| > 1$, or $\|\lambda^{-1}\mu_K^{-1}\kappa_K\| < 1$, i.e., $\|\lambda\kappa_K^{-1}\mu_K\| > 1$, condition, Eq. (42) or Eq. (A7) is valid.

3) In general, the effect of the higher-power series terms (higher than λ^3 or λ^{-3}) is very small. This shows that the second-order approximation or, at the most, the third-order approximation solution, has sufficient accuracy in practical engineering design.

4) When Φ_k only contains two eigenvectors of the first repeated root λ_1, accurate results of all eigenvector derivatives can be found by using the PDF formula Eq. (49).

5) Like the method presented in Ref. 15, a merit of the method in this paper is that matrices $A_0, A_1, A_{-1}, A_2, A_{-2}, \ldots$ are not associated with λ; that is, they only need to be computed once for the calculation of many eigenvector derivatives.

Table 4 Eigenvector derivatives of the fourth repeated root ($NM1 = 0.412, NM2 = 0.205 \times 10^2$)

				SDF method based on Eq. (41)					
Exact value		PDF method		First-order approximation		Second-order approximation		Third-order approximation	
$Z'_{4.1}$	$Z'_{4.2}$	$Z'_{4.1}$	$Z'_{4.2}$	$Z'_{4.1}$	$Z'_{4.2}$	$Z'_{4.1}$	$Z'_{4.2}$	$Z'_{4.1}$	$Z'_{4.2}$
0.0	−0.11648D−2	0.0	−0.11648D−2	0.0	−0.21051D−2	0.0	−0.14300D−2	0.0	−0.11984D−2
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	−0.49689D−1	0.0	−0.49689D−1	0.0	−0.43447D−1	0.0	−0.47931D−1	0.0	−0.49467D−1
0.0	0.13836D−1	0.0	0.13836D−1	0.0	0.13399D−1	0.0	0.13713D−1	0.0	0.13820D−1
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	0.33253D−1	0.0	0.33253D−1	0.0	0.30230D−1	0.0	0.32397D−1	0.0	0.33143D−1
0.0	−0.10408D−1	0.0	−0.10408D−1	0.0	−0.10160D−1	0.0	−0.10338D−1	0.0	−0.10399D−1
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0	0.0	0
0.0	0.49631D−1	0.0	0.49631D−1	0.0	0.45024D−1	0.0	0.48328D−1	0.0	0.49464D−1
Z'_{ij}%		0	0	0	10.54	0	2.976	0	0.379

Table 5 Eigenvector derivatives of the fifth repeated root ($NM3 = 0.984, NM4 = 0.715 \times 10^5$)

		SDF method based on Eq. (A7)					
Exact value		First-order approximation		Second-order approximation		Third-order approximation	
$Z'_{5.1}$	$Z'_{5.2}$	$Z'_{5.1}$	$Z'_{5.2}$	$Z'_{5.1}$	$Z'_{5.2}$	$Z'_{5.1}$	$Z'_{5.2}$
0.0	−0.10416D−1	0.0	−0.13614D−1	0.0	−0.11286D−1	0.0	−0.10660D−1
0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0
0.0	0.18815	0.0	0.20155	0.0	0.19184	0.0	0.18919
0.0	−0.76025D−2	0.0	−0.71554D−2	0.0	−0.74971D−2	0.0	−0.75739D−2
0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0
0.0	0.12017	0.0	0.11003	0.0	0.11732	0.0	0.11936
0.0	−0.40559D−2	0.0	−0.33857D−2	0.0	−0.38443D−2	0.0	−0.39950D−2
0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0
0.0	−0.34793D−2	0.0	0.62870D−2	0.0	−0.70706D−3	0.0	−0.26964D−2
Z'_{ij}%		0	8.814	0	2.460	0	0.692

Table 6 Eigenvector derivatives of the sixth repeated root ($NM3 = 0.248, NM4 = 0.248 \times 10^6$)

		SDF method based on Eq. (A7)					
Exact value		First-order approximation		Second-order approximation		Third-order approximation	
$Z'_{6.1}$	$Z'_{6.2}$	$Z'_{6.1}$	$Z'_{6.2}$	$Z'_{6.1}$	$Z'_{6.2}$	$Z'_{6.1}$	$Z'_{6.2}$
0.0	−0.11916D−1	0.0	−0.11910D−1	0.0	−0.11914D−1	0.0	−0.11915D−1
0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0
0.0	0.83394D−1	0.0	0.83350D−1	0.0	0.83389D−1	0.0	0.83393D−1
0.0	−0.66436D−2	0.0	−0.66372D−2	0.0	−0.66437D−2	0.0	−0.66436D−2
0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0
0.0	−0.42268D−1	0.0	−0.42213D−1	0.0	−0.42264D−1	0.0	−0.42268D−1
0.0	0.39759D−2	0.0	0.39621D−2	0.0	0.39754D−2	0.0	0.39758D−2
0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0
0.0	−0.61654D−1	0.0	−0.61720D−1	0.0	−0.61658D−1	0.0	−0.61654D−1
Z'_{ij}%		0	0.872	0	0.0036	0	0.26D−3

VI. Conclusion

Two different dynamic flexibility formulas are developed based on the practical complete modal space technique. When both formulas are utilized to compute the derivatives of many eigenvectors, PDF and SDF methods provide good numerical results. The SDF method not only has satisfactory precision, but also has computationally higher efficiency with regard to the derivative of many eigenvectors. Evolution of the generalized inverse technique is also used in the derivation of the equations. Note that the dynamic flexibility expressions in this paper can be applied to many other technical areas, such as dynamic response, dynamic structural modification, and synthesis of substructures.

Appendix: PDF and SDF Methods for Calculation of Highest-Order Eigenvector Derivative

The dynamic flexibility formulas used to compute the derivative of the highest-order eigenvector are developed. Also, the concept of practical complete modal space presented in Refs. 13 and 14 is

utilized in the manner of reversed direction. Using the concept presented in Refs. 13 and 14, an equivalent lower-order mode Ψ_k of Φ_k is formed as follows: First compute

$$\bar{\Psi}_k = \Phi_k \Phi_k^T M = I - \Phi_h \Phi_h^T M, \qquad \left(\bar{\Psi}_k \in R^{n,n}\right) \tag{A1}$$

Clearly, $\text{rank}(\bar{\Psi}_k) = k = n - h$. Then, similarly, eliminating h-dependent columns from the matrix $\bar{\Psi}_k$ obtains $\tilde{\Psi}_k$ ($\in R^{n,k}$). Thus, other PDF and SDF formulas corresponding to Eqs. (15) and (34) may be written, respectively, as

$$f(\lambda) = \Phi_h(\Omega_h - \lambda I)^+ \Phi_h^T + \Psi_k(\kappa_k - \lambda\mu_k)^{-1}\Psi_k^T \tag{A2}$$

$$f_s(\lambda) = \Phi_h(\Omega_h - \lambda I)^+ \Phi_h^T + \lambda^{-1}\Psi_k A_{-1}\Psi_k^T + \lambda^{-2}\Psi_k A_{-2}\Psi_k^T + \cdots \tag{A3}$$

where

$$\kappa_k = \Psi_k^T K \Psi_k, \qquad \mu_k = \Psi_k^T M \Psi_k \tag{A4}$$

Now, Z is included in Φ_h. The governing equations for solving matrices $A_{-1}, A_{-2}, \ldots$ are shown as follows:

$$\mu_k A_{-1} = -I \tag{A5}$$

$$\mu_k A_{-P} = \kappa_k A_{-(p-1)}, \qquad p \geq 2 \tag{A6}$$

The condition for convergence of the power series as shown in Eq. (A3) is $\|\lambda^{-1}\mu_k^{-1}\kappa_k\| < 1$. Another perturbed SDF formula corresponding to Eq. (42) is

$$f_s(\tilde{\lambda}) = \Phi_h(\Omega_h - (1+\eta)\lambda I)^{-1}\Phi_h^T + \sum_{p=1}^{s}(1+\eta)^{-p}\lambda^{-p}\Psi_k A_{-p}\Psi_k^T \tag{A7}$$

Obviously, the $\bar{f}_s(\tilde{\lambda})$ formula that is closer to $f_s(\lambda)$ shall be

$$\bar{f}_s(\tilde{\lambda}) = (I - ZZ^T M) f_s(\tilde{\lambda}) \tag{A8}$$

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